
The Presheaf Model of Type Theory in Type Theory

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Preliminaries

Type theory

- Simply typed λ -calculus: propositional logic

Type theory

- Simply typed λ -calculus: propositional logic
- Dependent type theory: first-order and higher logic

Type theory

 $e ::= x$ $| e_1 e_2 | \lambda(x : T).e$ $| (x : A) \rightarrow B$ $| (e_1, e_2) | \pi_1 e | \pi_2 e$ $| (x : A) \times B$ $| \text{Type}_\ell$

Type theory

A type $a \equiv b$ for $a, b : A$

Univalence

- Types are ∞ -groupoids...

Univalence

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- ...so types are spaces?!

Univalence

- Types are ∞ -groupoids...
- ...so types are spaces?!
- Add univalence on top of that:

$$(A \simeq B) \simeq (A \equiv B)$$

h-levels

$$\text{isContr } A = (a : A) \times ((x : A) \rightarrow x \equiv a)$$

h-levels

$$\text{isContr } A = (a : A) \times ((x : A) \rightarrow x \equiv a)$$

$$\text{isOfHLevel } 0 \ A = \text{isContr } A$$

$$\text{isOfHLevel } (\text{suc } n) \ A = (x \ y : A) \rightarrow \text{isOfHLevel } n \ (x \equiv y)$$

Cubical type theory

Univalence is a theorem!

Agda

- Proof assistant

Agda

- Proof assistant
- No tactics

Cubical Agda

Cubical Agda

Code:

```
funExt : (f g : (x : A) → B x) → ((x : A) → f x ≡ g x) → f ≡ g
funExt f g p i x = p x i
```

Internal models

Categories with Families

Definition 1 Category with Families (CwF)

A *category with families* is a category $\mathbf{Ctx}$ of *contexts* and *substitutions*, equipped with

- a terminal object $\langle \rangle : \mathbf{Ctx}$, the *empty context*
- a presheaf $\mathbf{Ty} : \widehat{\mathbf{Ctx}}$ of *types*
- a presheaf $\mathbf{Tm} : \int \mathbf{Ty}$ of *terms*

such that for all $\Gamma, A \in \int \mathbf{Ty}$, the following presheaf is representable:

$$\mathbf{Ext}(\Gamma, A) : \mathbf{Ctx}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

$$\Delta \mapsto (\sigma : \Delta \rightarrow \Gamma) \times \mathbf{Tm} \Delta (A[\sigma])$$

$$\tau \mapsto (j, i) \mapsto j \circ \tau, i[\tau]$$

Categories with Families

Theorem 1

The representability of $\text{Ext } (\Gamma, A)$ is equivalent to requiring

- an object $\Gamma \triangleright A \in \text{Ctx}$
- $p : \Gamma \triangleright A \rightarrow \Gamma$
- $q : \text{Tm } (\Gamma \triangleright A)(A[p])$
- that the following function is an equivalence:

$$\langle p, q \rangle : (\tau : \Delta \rightarrow \Gamma \triangleright A) \rightarrow (\sigma : \Delta \rightarrow \Gamma) \times \text{Tm } \Delta (A[\sigma])$$

$$\tau \mapsto p \circ \tau, q[\tau]$$

Formalizing CwFs

```

record CCwF (C : Category) : Type where
  field
    ⟨⟩ : Terminal C
    Ty : Presheaf C
    Tm : Presheaf (∫ Ty)
    _▷_ : (Γ : Ctx) (A : Ty[ Γ ]) → Ctx
    p    : {Γ : Ctx} {A : Ty[ Γ ]} → Γ ▷ A → Γ
    q    : {Γ : Ctx} {A : Ty[ Γ ]} → Tm[ Γ ▷ A , A [ p ]Ty ]

    ⟨p,q⟩ : {Γ Δ : Ctx} (A : Ty[ Γ ])
      → (Δ → Γ ▷ A) → Σ[ σ ∈ Δ → Γ ] Tm[ Δ , A [ σ ]Ty ]
    ⟨p,q⟩ {Δ = Δ} A τ =
      p ∘ τ , subst (λ T → Tm[ Δ , T ]) (sym (funExt⁻ (Ty .F-seq p τ) A)) (q [ τ ]Tm)

  field
    ctxExtRepr : {Γ Δ : Ctx} (A : Ty[ Γ ]) → isEquiv (⟨p,q⟩ {Γ} {Δ} A)

```

Formalizing CwFs

```

record ACwF (C : Category): Type where
  field
    ⟨⟩ : Terminal C
    Ty  : (Γ : Ctx) → Type
    Tm  : (Γ : Ctx) (A : Ty Γ) → Type

    isSetTy : (Γ : Ctx) → isSet (Ty Γ)
    isSetTm : (Γ : Ctx) (A : Ty Γ) → isSet (Tm Γ A)

    -- substitutions
    _[_]Ty : (A : Ty Γ) (σ : Δ → Γ)
             → -----
                Ty Δ
    [id]Ty : (A : Ty Γ)
             → -----
                A [ id ]Ty ≡ A
    [][_]Ty : (A : Ty Γ) (σ' : Θ → Δ) (σ : Δ → Γ)

```

```

→ -----
      A [ σ ∘ σ' ]Ty ≡ (A [ σ ]Ty) [ σ' ]Ty
_[_]Tm : {A : Ty Γ} (a : Tm Γ A) (σ : Δ → Γ)
→ -----
      Tm Δ (A [ σ ]Ty)
[id]Tm : {A : Ty Γ} (a : Tm Γ A)
→ -----
      PathP (λ i → Tm Γ ([id]Ty A i)) (a [ id ]Tm) a
[[]]Tm : {A : Ty Γ} (a : Tm Γ A) (σ' : θ → Δ) (σ : Δ → Γ)
→ -----
      PathP (λ i → Tm θ ([[]]Ty A σ' σ i))
              (a [ σ ∘ σ' ]Tm)
              (a [ σ ]Tm [ σ' ]Tm)

-- context extension
_▷_ : (Γ : Ctx) (A : Ty Γ) → Ctx
p : {A : Ty Γ} → Γ ▷ A → Γ
q : {A : Ty Γ} → Tm (Γ ▷ A) (A [ p ]Ty)

```

-- now, a bunch of coherence laws

$_+ : \{A : \text{Ty } \Gamma\} (\sigma : \Delta \rightarrow \Gamma)$

$\rightarrow \text{-----}$

$\Delta \triangleright A [\sigma] \text{Ty} \rightarrow \Gamma \triangleright A$

$\langle _ \rangle : \{A : \text{Ty } \Gamma\} (a : \text{Tm } \Gamma A)$

$\rightarrow \text{-----}$

$\Gamma \rightarrow (\Gamma \triangleright A)$

$\langle \rangle^\circ : \{A : \text{Ty } \Gamma\} (a : \text{Tm } \Gamma A) (\sigma : \Delta \rightarrow \Gamma)$

$\rightarrow \text{-----}$

$\langle a \rangle^\circ \sigma \equiv \sigma^+ \circ \langle a [\sigma] \text{Tm} \rangle$

$p^+ \circ \langle q \rangle \equiv \text{id} : \{A : \text{Ty } \Gamma\}$

$\rightarrow \text{-----}$

$p^+ \circ \langle q \rangle \equiv \text{id } \{\Gamma \triangleright A\}$

$\circ^+ : \{A : \text{Ty } \Gamma\} (\sigma' : \theta \rightarrow \Delta) (\sigma : \Delta \rightarrow \Gamma)$

$\rightarrow \text{-----}$

$\text{PathP } (\lambda i \rightarrow \theta \triangleright [][\text{Ty } A \sigma' \sigma i \rightarrow \Gamma \triangleright A]) ((\sigma \circ \sigma')^+) (\sigma^+ \circ \sigma'^+)$

$\text{id}^+ : \{A : \text{Ty } \Gamma\}$

$\rightarrow \text{-----}$

$\text{PathP } (\lambda i \rightarrow \Gamma \triangleright [\text{id}] \text{Ty } A i \rightarrow \Gamma \triangleright A) (\text{id}^+) \text{id}$

$$p \circ^+ : \{A : \text{Ty } \Gamma\} (\sigma : \Delta \rightarrow \Gamma)$$

$$\rightarrow \text{-----}$$

$$p \{A = A\} \circ \sigma^+ \equiv \sigma \circ p$$

$$[p][^+] \text{Ty} : \{A : \text{Ty } \Gamma\} (B : \text{Ty } \Gamma) (\sigma : \Delta \rightarrow \Gamma)$$

$$\rightarrow \text{-----}$$

$$B [p \{A = A\}] \text{Ty} [\sigma^+] \text{Ty} \equiv B [\sigma] \text{Ty} [p] \text{Ty}$$

$$q[^+] \text{Tm} : \{A : \text{Ty } \Gamma\} (\sigma : \Delta \rightarrow \Gamma)$$

$$\rightarrow \text{-----}$$

$$\text{PathP } (\lambda i \rightarrow \text{Tm } (\Delta \triangleright A [\sigma] \text{Ty}) ([p][^+] \text{Ty } A \sigma i)) (q [\sigma^+] \text{Tm}) q$$

$$p \circ \langle \rangle \equiv \text{id} : \{A : \text{Ty } \Gamma\} (a : \text{Tm } \Gamma A)$$

$$\rightarrow \text{-----}$$

$$p \circ \langle a \rangle \equiv \text{id}$$

$$[p][\langle \rangle] \text{Ty} : \{A : \text{Ty } \Gamma\} (B : \text{Ty } \Gamma) (a : \text{Tm } \Gamma A)$$

$$\rightarrow \text{-----}$$

$$B [p] \text{Ty} [\langle a \rangle] \text{Ty} \equiv B$$

$$q[\langle \rangle] \text{Tm} : \{A : \text{Ty } \Gamma\} (a : \text{Tm } \Gamma A)$$

$$\rightarrow \text{-----}$$

$$\text{PathP } (\lambda i \rightarrow \text{Tm } \Gamma ([p][\langle \rangle] \text{Ty } A a i)) (q [\langle a \rangle] \text{Tm}) a$$

Formalizing CwFs

Theorem 2



Given a category C , the types $\mathbf{CCwF} \ C$ and $\mathbf{ACwF} \ C$ are equivalent.

Type structures

Definition 2 Σ -structure



A Σ -structure is an assignment of $A : \text{Ty } \Gamma$, $B : \text{Ty } (\Gamma \triangleright A)$ to some $\Sigma AB : \text{Ty } \Gamma$, that is natural in A and B . There must be a natural isomorphism $\text{Tm } \Gamma (\Sigma AB) \simeq (a : \text{Tm } \Gamma A) \times \text{Tm } \Gamma B[\langle a \rangle]$.

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Definition 3 Π -structure



A Π -structure is an assignment of $A : \text{Ty } \Gamma$, $B : \text{Ty } (\Gamma \triangleright A)$ to a type $\Pi AB : \text{Ty } \Gamma$ that is natural in A and B , with a natural isomorphism $\text{Tm } \Gamma (\Pi AB) \simeq \text{Tm } (\Gamma \triangleright A) B$.

Type structures

Definition 2 Σ -structure



A Σ -structure is an assignment of $A : \text{Ty } \Gamma$, $B : \text{Ty } (\Gamma \triangleright A)$ to some $\Sigma AB : \text{Ty } \Gamma$, that is natural in A and B . There must be a natural isomorphism $\text{Tm } \Gamma (\Sigma AB) \simeq (a : \text{Tm } \Gamma A) \times \text{Tm } \Gamma B[\langle a \rangle]$.

Definition 3 Π -structure



A Π -structure is an assignment of $A : \text{Ty } \Gamma$, $B : \text{Ty } (\Gamma \triangleright A)$ to a type $\Pi AB : \text{Ty } \Gamma$ that is natural in A and B , with a natural isomorphism $\text{Tm } \Gamma (\Pi AB) \simeq \text{Tm } (\Gamma \triangleright A) B$.

Definition 4 Eq-structure



An Eq-structure is an assignment of $A : \text{Ty } \Gamma$, $a, b : \text{Tm } \Gamma A$ to a type $\text{Eq } A a b : \text{Ty } \Gamma$, natural in all variables, and with a natural isomorphism $\text{Tm } \Gamma (\text{Eq } A a b) \simeq a \equiv b$.

The set and presheaf models

Sets in HoTT

The type of h-sets is not an h-set :(

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$$V^\infty = W_{A: \text{Type}} A$$

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Iterative sets to the rescue!

$$V^\infty = W_{A : \mathbf{Type}} A$$

$$\text{is0Set} : V^\infty \rightarrow \mathbf{Type}$$

$$\text{is0Set} (\text{sup } A f) = \text{isEmbedding } f \times ((a : A) \rightarrow \text{is0Set } (f a))$$

$$V^0 : \mathbf{Type}$$

$$V^0 = (X : V^\infty) \times \text{is0Set } X$$

Sets in HoTT

Theorem 3 H. R. Gylterud [1]

For $x, y : V^0$, we have

$$x \equiv y \simeq (z : V^0) \rightarrow (z \in x \leftrightarrow z \in y)$$

Tarski universes

Tarski universes

```
record TarskiUniverse-Base : Type where
  field
    U : Type
    El : U → Type
    isSetU : isSet U
    isSetEl : (x : U) → isSet (El x)
```

Tarski universes

```
record TarskiUniverse-Base : Type where
  field
    U : Type
    El : U → Type
    isSetU : isSet U
    isSetEl : (x : U) → isSet (El x)
```

Theorem 4



V^0 is a Tarski universe.

Tarski universes

```
record TarskiUniverse-Unit : Type where
  field
    Unit : U
    isContrElUnit : isContr (El Unit)
record TarskiUniverse-Sig : Type where
  field
    Sig : (A : U) → (El A → U) → U
    SigIso : (A : U) (B : El A → U)
              → Iso (El (Sig A B)) (Σ[ x ∈ El A ] El (B x))
```

Tarski universes

- Iterative sets
- Finite sets
- Cumulative hierarchy
- Inductive-recursive universe

The set model

Theorem 5



For every Tarski universe U , $\mathcal{U}$ can be equipped with an ACwF structure $\mathbf{U}\text{-CwF } \mathcal{U}$, where

- $\langle \rangle = \mathbf{Unit}$
- $\mathbf{Ty } \Gamma = \mathbf{El } \Gamma \rightarrow U$
- $\mathbf{Tm } \Gamma A = (\rho : \mathbf{El } \Gamma) \rightarrow \mathbf{El } (A\rho)$
- $\Gamma \triangleright A = \mathbf{Sig } \Gamma A$

The set model

Theorem 5



For every Tarski universe U , U can be equipped with an ACwF structure $\mathbf{U}\text{-CwF } U$, where

- $\langle \rangle = \text{Unit}$
- $\text{Ty } \Gamma = \text{El } \Gamma \rightarrow U$
- $\text{Tm } \Gamma A = (\rho : \text{El } \Gamma) \rightarrow \text{El } (A\rho)$
- $\Gamma \triangleright A = \text{Sig } \Gamma A$

Theorem 6



- $\mathbf{U}\text{-CwF } U$ has a Σ -structure
- If U has a Π -structure, $\mathbf{U}\text{-CwF } U$ has a Π -structure
- If U has an Eq-structure, $\mathbf{U}\text{-CwF } U$ has an Eq-structure

The presheaf model

Theorem 7



There is an ACwF on $\hat{C}$ defined by

- $\langle \rangle$ is the constant presheaf with value `Unit`
- $\text{Ty } \Gamma$ is the set of presheaves $(\int \Gamma)^{\text{op}} \rightarrow \mathcal{U}$
- $\text{Tm } \Gamma A$ is the set of natural transformations from the unit type to A , where the unit type in context Γ is the constant presheaf with value `Unit`
- Context extension is defined by $(\Gamma \triangleright A)I := (\rho : \text{El } (\Gamma I)) \times A(I, \rho)$.

The presheaf model

Theorem 8



The CwF on $\hat{C}$ has a Σ -structure.

The presheaf model

$$\begin{aligned}
 (\Pi AB)(I, \rho) &:= \{w : (J : \mathcal{C})(f : J \rightarrow I)(a : A(J, \Gamma f \rho)) \rightarrow B(J, (\Gamma f \rho, a)) \\
 &\quad | \ \forall g : K \rightarrow J, Bg(wfa) = w(f \cdot g)(Aga) \\
 &\quad \} \\
 &= \int_{f:J \rightarrow I} (a : A(J, \Gamma f \rho)) \rightarrow B(J, (\Gamma f \rho, a))
 \end{aligned}$$

Conclusion & Future Work

Bibliography

- [1] H. R. Gylterud, “From Multisets to Sets in Homotopy Type Theory,” *The Journal of Symbolic Logic*, vol. 83, no. 3, pp. 1132–1146, Sep. 2018, doi: 10.1017/jsl.2017.84.